

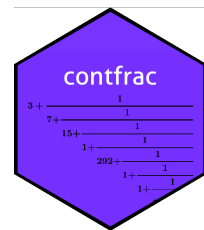
Continued fractions in R: introducing the `confrac` package

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Abstract

Here I introduce the `confrac` package, for manipulating continued fractions.

Keywords: Continued fractions.



1. Overview

A *continued fraction* is an expression of the form

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}} \quad (1)$$

provided that the sequence

$$f_0 = a_0, \quad f_1 = a_0 + \frac{1}{a_1}, \quad f_2 = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, \dots \quad (2)$$

converges. This is written $[a_0; a_1, a_2, \dots]$ for convenience. A *generalized continued fraction* is of the form

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ddots}}} \quad (3)$$

but for reasons of typographical convenience this would usually be written

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \frac{a_4}{b_4 +} \dots \quad (4)$$

Continued fractions are an important branch of mathematics (Lorentzen and Waadeland 2008) and are useful in a range of numerical disciplines (Hankin 2017), and I will give some numerical examples here.

Continued fractions furnish, in a well-defined sense, the “best possible” rational approximations to a given real number (Hall and Knight 1891). For example, we may take the continued fraction

$$\pi = 3 + \frac{1}{7 +} \frac{1}{15 +} \frac{1}{1 +} \frac{1}{291 +} \frac{1}{1 +} \dots \quad (5)$$

to deduce that $\frac{22}{7}$ and $\frac{355}{113}$ are close to π , being the second and fourth convergents respectively. The R idiom for this would use `as_cf()` which uses numerical methods to calculate the first few terms:

```
> library("contfrac")
> as_cf(pi, n=7) # calculate the first 7 terms
```

```
[1] 3 7 15 1 292 1 1
```

We can expand the first few terms of the series and verify that the series does converge to π with function `convergents()`:

```
> (jj <- convergents(as_cf(pi, n=7)))
```

```
$A
```

```
[1] 3 22 333 355 103993 104348 208341
```

```
$B
```

```
[1] 1 7 106 113 33102 33215 66317
```

```
> jj$A/jj$B - pi
```

```
[1] -1.415927e-01 1.264489e-03 -8.321963e-05 2.667642e-07 -5.778906e-10
```

```
[6] 3.316281e-10 -1.223563e-10
```

(the signs alternate and decrease in absolute value). Because the package uses standard IEEE precision, the continued fraction expansion for any non-rational number will eventually succumb to rounding error. We may investigate this using quadratic surds whose expansions are repeating patterns; the most famous would be that of $\phi = \frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, 1, \dots]$:

```
> (phi <- (1+sqrt(5))/2)
```

```
[1] 1.618034
```

