Continued fractions in **R**: introducing the **contfrac** package

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Abstract

Here I introduce the **contfrac** package, for manipulating continued fractions.

Keywords: Continued fractions.

1. Overview

A *continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \tag{1}$$

provided that the sequence

$$f_0 = a_0, \qquad f_1 = a_0 + \frac{1}{a_1}, \qquad f_2 = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, \ldots \tag{2}$$

converges. This is written $[a_0; a_1, a_2, \ldots]$ for convenience. A *generalized continued fraction* is of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \tag{3}$$

but for reasons of typographical convenience this would usually be written

$$b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\frac{a_4}{b_4+\cdots}. \tag{4}$$

Continued fractions are an important branch of mathematics (Lorentzen and Waadeland 2008) and are useful in a range of numerical disciplines (Hankin 2017), and I will give some numerical examples here.

Continued fractions furnish, in a well-defined sense, the "best possible" rational approximations to a given real number (Hall and Knight 1891). For example, we may take the continued fraction

$$\pi = 3 + \frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{291+}\frac{1}{1+\cdots} \tag{5}$$

to deduce that $\frac{22}{7}$ and $\frac{355}{113}$ are close to π, being the second and fourth convergents respectively. The R idiom for this would use `as_cf()` which uses numerical methods to calculate the first few terms:

```
> library("contfrac")
> as_cf(pi, n=7)  # calculate the first 7 terms

[1]   3   7  15   1 292   1   1
```

We can expand the first few terms of the series and verify that the series does converge to π with function `convergents()`:

```
> (jj <- convergents(as_cf(pi,n=7)))

$A
[1]      3     22    333    355 103993 104348 208341

$B
[1]     1     7   106   113 33102 33215 66317

> jj$A/jj$B - pi

[1] -1.415927e-01  1.264489e-03 -8.321963e-05  2.667642e-07 -5.778906e-10
[6]  3.316281e-10 -1.223563e-10
```

(the signs alternate and decrease in absolute value). Because the package uses standard IEEE precision, the continued fraction expansion for any non-rational number will eventually succumb to rounding error. We may investigate this using quadratic surds whose expansions are repeating patterns; the most famous would be that of $\phi = \frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, 1, \ldots]$:

```
> (phi <- (1+sqrt(5))/2)

[1] 1.618034
```

```
> as_cf(phi,n=50)

 [1] 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
[39] 2 2 1 8 2 2 2 3 2 1 2 3

> rle(as_cf(phi,n=50))

Run Length Encoding
  lengths: int [1:10] 38 2 1 1 3 1 1 1 1 1
  values : num [1:10] 1 2 1 8 2 3 2 1 2 3
```

so in this case we can see that the first 38 terms are accurate.

Generalized continued fractions

The package provides functionality for generalized continued fractions, which give alternatives to Taylor or Maclaurin series for many functions. For example, we have

$$\tan(z) = \frac{z}{1-}\,\frac{z^2}{3-}\,\frac{z^2}{5-}\,\frac{z^2}{7-}\,\frac{z^2}{9-}\cdots \qquad z/\pi + 1/2 \notin \mathbb{Z} \tag{6}$$

and we can show this series in R as follows:

```
> tan_cf <- function(z,n=20){GCF(c(z,rep(-z^2,n-1)),seq(from=1,by=2,len=n))}
> z <- 1+1i
> tan(z) - tan_cf(z) # should be small

[1] 1.110223e-16+4.440892e-16i
```

References

Hall HS, Knight SR (1891). *Higher algebra*. Macmillan.

Hankin RKS (2017). "Numerical evaluation of the Gauss hypergeometric function with the **hypergeo** package." *The R Journal*, **7**(2), 81–88.

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